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GLOBALLY IRREDUCIBLE REPRESENTATIONS OF $SL_3(q)$ AND $SU_3(q)$

BY

PHAM HUU TIEP

Department of Mathematics, University of Florida
Gainesville, FL 32611, USA
e-mail: tiep@math.ufl.edu

ABSTRACT

The notion of **globally irreducible representations** of finite groups was introduced by B. H. Gross, in order to explain new series of Euclidean lattices discovered by N. Elkies and T. Shioda using Mordell–Weil lattices of elliptic curves. In this paper we classify all globally irreducible representations coming from projective complex representations of the finite simple groups $PSL_3(q)$ and $PSU_3(q)$. The main result is that these representations are essentially those discovered by Gross.

1. Introduction

In [Tho] J. G. Thompson initiated the study of pairs (G, Λ), where G is a finite group and Λ a G-module, which is a free $\mathbb{Z}$-module of finite rank > 1, that satisfy the following condition:

(1) $\Lambda/p\Lambda$ is an irreducible $\mathbb{F}_pG$-module for all primes p.

At present there are known only a few examples of such pairs (cf. e.g. [Gow]).

A generalization of (1)—the notion of **globally irreducible representations** (abbreviated as GIR's)—was introduced by B. H. Gross [Gro] in order to explain new interesting series of Euclidean lattices discovered by N. Elkies and T. Shioda [Elk], [Shi] using Mordell–Weil lattices of elliptic curves. For the reader's convenience we recall the precise definition of GIR's.

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Definition 1.1 [Gro]: Let G be a finite group, V a finite dimensional $\mathbb{Q}G$-module. Assume that

(i) The $\mathbb{R}G$-module $V \otimes_{\mathbb{Q}} \mathbb{R}$ is irreducible.

This condition implies that the commuting algebra,

$$\mathbb{K} = \mathrm{End}_{\mathbb{Q}G}(V) = \{\varphi \in \mathrm{End}_{\mathbb{Q}}(V) | \; \forall g \in G, \varphi g = g\varphi\},$$

is either $\mathbb{Q}$, or an imaginary quadratic field, or a definite quaternion algebra. Fix a maximal order (i.e. a maximal subring which is a free $\mathbb{Z}$-module) R in $\mathbb{K}$ and choose an RG-lattice in V, that is, a free $\mathbb{Z}$-submodule Λ in V of rank $= \dim_{\mathbb{Q}}(V)$ which is stable under the multiplication by R and the action of G. For each maximal two-sided ideal $\wp$ of R, we define the reduced representation $V_\wp = \Lambda/\wp\Lambda$ of G over $k_\wp = R/\wp$. If, in addition to (i), V satisfies the following condition:

(ii) For all maximal two-sided ideals $\wp$ of R, the representation $V_\wp$ of G is irreducible over $k_\wp$,

then V is said to be **globally irreducible**.

We shall frequently use the following necessary condition for global irreducibility (cf. [Tiep 1, Proposition 2.8]). Here and below, the words "ψ leads to a GIR ..." mean that $\psi \in \mathrm{Irr}(G)$ is an irreducible constituent of $\chi|_G$, where χ is the character of a certain GIR V for some larger group H with $G \lhd H$.

PROPOSITION 1.2 [Tiep 1]: *Let G be a finite group and $\psi \in \mathrm{Irr}(G)$. Assume that ψ leads to a GIR. Then for every rational prime r, there is an absolutely irreducible Brauer character ρ of G such that $\psi \equiv \rho_1 + \cdots + \rho_s (\mathrm{mod} r)$, where $\rho_1, \ldots, \rho_s$ are some conjugates (over $\overline{\mathbb{F}_r}$ and under $\mathrm{Aut}(G)$) of ρ. In particular, all irreducible constituents of ψ mod r have the same degree, and this degree divides $\psi(1)$.*

For a survey on the currently known classes of GIR's as well as the corresponding lattices the reader is referred to [Gro], [Tiep 1]. All GIR's coming from projective irreducible representations of the finite simple groups $L_2(q)$ and ${}^2B_2(q)$ are determined in [Tiep 1]. The case $B_2(q)$ has been treated in [Tiep 2].

The goal of this paper is to prove the following theorems, which determine all GIR's coming from projective irreducible representations of the finite simple groups $L_3(q) = \mathrm{PSL}_3(q)$ and $\mathrm{PSU}_3(q)$. The ordinary irreducible characters of $\mathrm{SL}_3(q)$ and $\mathrm{SU}_3(q)$ are determined by W. Simpson and J. S. Frame in [SiF] (cf. also [Geck]), and we shall keep the notation for irreducible characters of $\mathrm{SL}_3(q)$ and $\mathrm{SU}_3(q)$ of this paper. Furthermore, the notation for finite simple groups, their coverings and extensions are taken from [ATLAS].

THEOREM 1.3: *Suppose that V is a faithful GIR for a finite group H with character χ, and that H has a normal subgroup G which is a covering group of the finite simple group $L = A_2(q) = \mathrm{PSL}_3(q)$, where $q = p^f$, p is a prime. Then $p = 2$ and one of the following holds.*

(i) $q = 2$, $G = \mathrm{SL}_3(2) \simeq \mathrm{PSL}_2(7)$, $\chi_G = \kappa(\lambda + \bar{\lambda})$, *where* $\kappa \in \mathbb{N}$, $\lambda \in \mathrm{Irr}(G)$, $\deg \lambda = 3$.

(ii) $q = 4$, $G = 4_1 \cdot L_3(4)$, $\chi_G = \kappa(\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4)$, *where* $\kappa \in \mathbb{N}$ *and* α_i, $i = 1, 2, 3, 4$, *are the only irreducible characters of* G *of degree* 8.

(iii) $q = 4$, $G = 6 \cdot L_3(4)$, $\chi_G = \kappa(\beta_1 + \beta_2)$, *where* $\kappa \in \mathbb{N}$ *and* β_i, $i = 1, 2$, *are the only irreducible characters of* G *of degree* 6.

(iv) $q = 4$, $G = 12_1 \cdot L_3(4)$, $\chi_G = \kappa(\gamma_1 + \gamma_2 + \cdots + \gamma_8)$, *where* $\kappa \in \mathbb{N}$ *and* γ_i, $i = 1, \ldots, 8$, *are the only irreducible characters of* G *of degree* 24.

Conversely, there exist GIR's of H *with* $(G, \chi|_G)$ *as described in* (i), (ii), (iii), *and* $(H, \kappa) = (G, 1)$ *in case* (i), $(H, \kappa) = (G \cdot 2_1, 1)$ *in case* (ii), *and* $(H, \kappa) = (G \cdot 2_2, 2)$ *in case* (iii).

As can be seen from Theorem 1.3, all GIR's coming from projective irreducible representations of the finite simple group $L_3(q)$ are related to the exceptional behavior of the Schur multiplier of L when $q = 2, 4$. It is well known that $\mathrm{Mult}(L)$ is equal to $\mathbb{Z}_{\gcd(3,q-1)}$ if $q \neq 2, 4$, $\mathbb{Z}_2$ if $q = 2$, and $\mathbb{Z}_4 \times \mathbb{Z}_{12}$ if $q = 4$.

The 6-dimensional $\mathrm{SL}_3(2)$-lattice Λ coming up in 1.3 (i) arises from the theta polarization of the Jacobian of the complex curve X with equation $xy^3 + yz^3 + zx^3 = 0$ (cf. [Gro]). This lattice has the following invariants: $\det \Lambda = 7^3$, $\mathrm{Aut}(\Lambda) = 2 \times \mathrm{PGL}_2(7)$, $\min \Lambda = 4$.

The 24-dimensional $6 \cdot L_3(4)$-lattice arising in 1.3 (iii) is the famous Leech lattice.

At present we do not know much about the 32-dimensional $4 \cdot L_3(4) \cdot 2$-lattice arising in 1.3 (ii). We conjecture that it has determinant 10^{16}.

THEOREM 1.4: *Suppose that V is a faithful GIR for a finite group H with character χ, and that H has a normal subgroup G which is a covering group of the finite simple group $L = {}^2A_2(q) = \mathrm{PSU}_3(q)$, where $q = p^f > 2$, p is a prime. Then $f \leq 2$, $G = L$, $\chi|_G = 2\kappa\chi_{q^2-q}$, where $\kappa \in \mathbb{N}$, and χ_{q^2-q} is the unipotent cuspidal character of G of degree $q^2 - q$. Conversely, if $f \leq 2$ then G has a globally irreducible representation with character $2\chi_{q^2-q}$.*

In particular, Theorem 1.4 shows that all GIR's of $\mathrm{SU}_3(q)$ are exhausted by those discovered by Gross in [Gro]. A Mordell–Weil model for the lattices Λ in the corresponding globally irreducible modules has been obtained by Elkies (cf.

[Gro]). Let E be a supersingular elliptic curve over $\mathbb{F}_{q^2}$ with Frobenius endomorphism equal to $-q$ and $\mathrm{End}(E)$ a maximal order in the quaternion algebra ramified at p and ∞. Let X be the Fermat curve of exponent $q+1$ in characteristic p and J_X denote the Jacobian of X. Then $\Lambda = \mathrm{Hom}_{\mathbb{F}_{q^2}}(J_X, E)$. (Clearly, the group $\mathrm{PU}_3(q)$ acts on X and so on Λ.) As is shown in [Gro], $\det \Lambda$ is 1 if $q = p^2$, and $p^{p(p-1)}$ if $q = p$. These lattices have been given a detailed investigation by N. Dummigan [Dum 1], [Dum 2].

A crucial ingredient of our proofs is Jantzen's formula on reduction modulo p of Deligne–Lusztig characters [Jan 1], [Jan 2], which enables one to exclude the majority of the irreducible characters of $\mathrm{SL}_3(q)$ and $\mathrm{SU}_3(q)$. Then one applies the results in [Geck], [Jam], [JLPW] on modular representations in non-defining characteristics to deal with the remaining characters.

The remaining finite groups of Lie type of rank 2 are considered in forthcoming papers.

2. Preliminaries

Our proof is heavily based on the representation theory of Chevalley groups in their defining characteristic (see e.g. [Jan 1]). Let $\mathcal{G}$ be the connected, simply connected, and semisimple algebraic group of type A_2 defined over $\mathbb{F}_p$; $k = \overline{\mathbb{F}_p}$, $T = \{\mathrm{diag}(a,b,c) |\ a,b,c \in k, abc = 1\}$ a maximal torus in $\mathcal{G}$, defined and split over $\mathbb{F}_p$, W the Weyl group of $\mathcal{G}$ with respect to T. We realize a basis S for the root system of $\mathcal{G}$ with respect to T in standard way: $S = \{\alpha_1 = e_1 - e_2, \alpha_2 = e_2 - e_3\}$, where $\{e_1, e_2, e_3\}$ is an orthonormal basis of $\mathbb{R}^3$. Then the fundamental weights $\omega_i = \omega_{\alpha_i}$ corresponding to $\alpha_i \in S$ are equal to

$$\omega_1 = \frac{1}{3}(2e_1 - e_2 - e_3) = \frac{1}{3}(2\alpha_1 + \alpha_2),\ \omega_2 = \frac{1}{3}(e_1 + e_2 - 2e_3) = \frac{1}{3}(\alpha_1 + 2\alpha_2).$$

Set $\rho = \omega_1 + \omega_2 = e_1 - e_3$. The symbols $X(T)$ and $X(T)_+$ denote the group of characters on T and the set of dominant weights, respectively. For $f \in \mathbb{N}$ denote

$$X_f(T) = \left\{ \sum_{\alpha \in S} m_\alpha \omega_\alpha |\ 0 \le m_\alpha < p^f \text{ for all } \alpha \in S \right\}.$$

We shall use the following order relation $\le$ on $X(T)$: $\lambda \le \mu$ if and only if there are non-negative $m_\alpha \in \mathbb{Z}$ such that $\mu - \lambda = \sum_{\alpha \in S} m_\alpha \alpha$. In our particular case of type A_2, we write every weight $\lambda = r\omega_1 + s\omega_2$ in the form $\lambda = (r, s)$ and define the following parameters of λ:

$$(2)\quad |\lambda|_1 = 2r + s, \quad |\lambda|_2 = r + 2s, \quad [\lambda] = (r^2 + rs + s^2 + 3(r + s + 1))(\bmod 3p\mathbb{Z}).$$

Of course,

$$\lambda = r\omega_1 + s\omega_2 = \frac{2r+s}{3}\alpha_1 + \frac{r+2s}{3}\alpha_2.$$

Hence, for $\lambda, \mu \in X(T)$ we have

$$\lambda \geq \mu \Leftrightarrow \begin{cases} |\lambda|_1 - |\mu|_1 \in 3\mathbb{Z}, \\ |\lambda|_1 \geq |\mu|_1,\ |\lambda|_2 \geq |\mu|_2. \end{cases} \tag{3}$$

It is well known that for each $\lambda \in X(T)_+$ there exists a unique simple $k\mathcal{G}$-module $L(\lambda)$ with highest weight λ. Furthermore, if $G = \mathcal{G}^F = \mathrm{SL}_3(q)$, $q = p^f$ ($F = Fr^f$ and Fr denotes the Frobenius endomorphism $x = (x_{ij}) \mapsto (x_{ij}^p)$ of $\mathcal{G}$), then every simple kG-module is isomorphic to exactly one $L(\lambda)$ with $\lambda = (r, s) \in X_f(T)$. The corresponding Brauer character will be denoted by $\varphi(\lambda) = \varphi(r, s)$. For $\lambda \in X(T)$ define the formal Brauer character $\zeta(\lambda)$ as follows. If $\lambda \in X(T)_+$ then $\zeta(\lambda)$ is the character of the so-called **Weyl module** $V(\lambda)$ with highest weight λ. If $\lambda \in X(T)$ and there exists $w \in W$ with $w\circ\lambda := w(\lambda+\rho)-\rho \in X(T)_+$, then $\zeta(\lambda) = \det(w)\zeta(w\circ\lambda)$; if not, then $\zeta(\lambda) = 0$. It is known that for $\lambda \in X(T)_+$ one has

$$\zeta(\lambda) = \varphi(\lambda) + \sum_{\mu \in X(T)_+, \mu < \lambda} d_{\lambda,\mu}\varphi(\mu), \tag{4}$$

where $d_{\lambda,\mu} \geq 0$. Moreover, the **linkage principle** states that if in (4), $d_{\lambda,\mu} > 0$, then λ and μ are **linked**, i.e. there exists $w \in W_p$ such that $\mu = w \circ \lambda$. Here, the affine Weyl group W_p is generated by the maps

$$s_{\alpha,l} \circ \lambda = s_\alpha \circ \lambda + lp\alpha,$$

where $\alpha \in S$, $l \in \mathbb{Z}$ and s_α denotes the usual reflection corresponding to α. Immediately we have the following statement:

LEMMA 2.1: *If $\lambda, \mu \in X(T)_+$ are linked, then $[\lambda] = [\mu]$.*

Proof: For brevity we write $|\lambda|^2$ instead of (λ, λ). The parameter $[\lambda]$ is in fact defined as $\frac{3}{2}|\lambda+\rho|^2 (\bmod 3p)$ (see (2) for the definition of $[\lambda]$, $|\lambda|_i$). Furthermore, for $\alpha \in S$, $l \in \mathbb{Z}$ one has

$$|s_{\alpha,l} \circ \lambda + \rho|^2 = |\lambda + \rho|^2 + l^2p^2|\alpha|^2 + 2lp(s_\alpha(\lambda+\rho), \alpha).$$

Observe that $|\alpha|^2 \in 2\mathbb{Z}$ and

$$(s_\alpha(\lambda+\rho), \alpha) = \left((\lambda+\rho) - \frac{2(\lambda+\rho,\alpha)}{(\alpha,\alpha)}, \alpha\right) = -(\lambda+\rho, \alpha) \in \mathbb{Z}.$$

Therefore, $[s_{\alpha,l} \circ \lambda] = [\lambda]$, as stated.

Corollary 2.2: *Suppose that* $\lambda = (r,s) \in X(T)_+$ *and* $\lambda' = (r',s') \in X(T)_+$. *Then* $\varphi(\lambda')$ *does not occur in* $\zeta(\lambda)$, *if any of the following conditions holds.*

(i) $|\lambda'|_i > |\lambda|_i$ *for some* $i = 1, 2$.

(ii) $[\lambda'] \neq [\lambda]$.

We are going to use Corollary 2.2 for $G = \mathrm{SL}_3(q)$, $\mathrm{SU}_3(q)$ with $q = p^f$, and $\lambda \in X_f(T)$. In this case (4) gives us all composition factors of the G-module $V(\lambda)$.

The following lemma is useful for computing $\zeta(\lambda) = \zeta(r,s)$ with $\lambda \notin X(T)_+$:

Lemma 2.3: *Suppose that* $r, s \in \mathbb{Z}$, *but* $\lambda \notin X(T)_+$.

(i) *If* (r,s) *belongs to the region* $(I) = \{(r,s) \mid r \leq -2, r+s \geq -1\}$, *then* $\zeta(r,s) = -\zeta(-r-2, r+s+1)$.

(ii) *If* (r,s) *belongs to the region* $(II) = \{(r,s) \mid r \leq -2, s \leq -2\}$, *then* $\zeta(r,s) = -\zeta(-s-2, -r-2)$.

(iii) *If* (r,s) *belongs to the region* $(III) = \{(r,s) \mid s \leq -2, r+s \geq -1\}$, *then* $\zeta(r,s) = -\zeta(r+s+1, -s-2)$.

(iv) *If* (r,s) *belongs to the region* $(A) = \{(r,s) \mid r+s \leq -3, r \geq 0\}$, *then* $\zeta(r,s) = \zeta(-r-s-3, r)$.

(v) *If* (r,s) *belongs to the region* $(B) = \{(r,s) \mid r+s \leq -3, s \geq 0\}$, *then* $\zeta(r,s) = \zeta(s, -r-s-3)$.

(vi) *Otherwise* $\zeta(\lambda) = 0$.

Proof: For example, we consider the case where there exists $w \in W$ with $\det(w) = -1$ and $w \circ \lambda \in X(T)_+$. For definiteness suppose that $w : e_1 \leftrightarrow e_2, e_3 \mapsto e_3$. Then

$$w(\omega_1) = \omega_2 - \omega_1,\ w(\omega_2) = \omega_2,$$
$$w(\lambda + \rho) - \rho = (-r-2)\omega_1 + (r+s+1)\omega_2,$$

and we come to conclusion (i). The remaining elements of W are treated in a similar way. ∎

Corollary 2.4: *If* $\lambda = (r,s) \notin X(T)_+$ *and* $r, s \geq -1$, *then* $\zeta(r,s) = 0$.

The dimension of the Weyl module $V(r,s)$ is given by the following formula (see e.g. [OnV]):

$$\deg \zeta(r,s) = \frac{(r+1)(s+1)(r+s+2)}{2}. \tag{5}$$

Furthermore, the decomposition of $V(\lambda) = V(r,s)$ with $\lambda \in X_1(T)$ into simple modules is described by the following statement:

LEMMA 2.5 [Jan 2]: *If $0 \le r, s \le p-1$ and either $r+s \le p-2$, or $r = p-1$, or $s = p-1$, then $\zeta(\lambda) = \varphi(\lambda)$. If $0 \le r, s \le p-2$ and $r+s \ge p-1$, then $\zeta(r,s) = \varphi(r,s) + \varphi(p-2-s, p-2-r)$.*

Using Lemma 2.5, formula (5) and Steinberg's tensor product theorem, one can determine the dimensions of all simple kG-modules $L(\lambda)$, $\lambda \in X_f(T)$.

3. The groups $\mathrm{SL}_3(q)$

3.1 A CRITERION FOR GLOBAL IRREDUCIBILITY. Throughout this section we suppose that $G = \mathcal{G}^F = \mathrm{SL}_3(q)$, $q = p^f$, and that a *nontrivial irreducible character $\chi \in \mathrm{Irr}(G)$ leads to a GIR*. By Proposition 1.2, for any arbitrary prime r all irreducible constituents of $\chi \bmod r$ are conjugate (over $\overline{\mathbb{F}_r}$ and under $\mathrm{Aut}(G)$). In particular, all irreducible constituents of $\chi \bmod r$ have the same degree, which divides $\deg\chi$. By Lemma 3.1 [Tiep 3], the action of $\mathrm{Out}(G)$ on $\mathrm{IBr}_p(G)$ is generated by the Frobenius endomorphism Fr and the graph automorphism $\omega_1 \leftrightarrow \omega_2$. Using this fact and following the proof of Corollary 2.7 [Tiep 2], one gets

COROLLARY 3.1: *Suppose that $\varphi(\lambda)$, $\varphi(\lambda')$ are irreducible constituents of $\chi \bmod p$, where $\lambda = (r,s)$ and $\lambda' = (r',s')$ belong to $X_f(T)$. Consider the p-adic decompositions of the integers r, s, r', s':*

$$r = \sum_{i=0}^{f-1} p^i r_i, \quad s = \sum_{i=0}^{f-1} p^i s_i, \quad r' = \sum_{i=0}^{f-1} p^i r'_i, \quad s' = \sum_{i=0}^{f-1} p^i s'_i.$$

Then there exists a cyclic permutation π of indices $0, 1, \ldots, f-1$ such that either

$$(r'_0, r'_1, \ldots, r'_{f-1}) = (r_{\pi(0)}, r_{\pi(1)}, \ldots, r_{\pi(f-1)}),$$
$$(s'_0, s'_1, \ldots, s'_{f-1}) = (s_{\pi(0)}, s_{\pi(1)}, \ldots, s_{\pi(f-1)}),$$

or

$$(r'_0, r'_1, \ldots, r'_{f-1}) = (s_{\pi(0)}, s_{\pi(1)}, \ldots, s_{\pi(f-1)}),$$
$$(s'_0, s'_1, \ldots, s'_{f-1}) = (r_{\pi(0)}, r_{\pi(1)}, \ldots, r_{\pi(f-1)}).$$

There exists an integer a, $0 \le a \le f-1$, such that either $r' \equiv p^a r \bmod (q-1)$ and $s' \equiv p^a s \bmod (q-1)$, or $r' \equiv p^a s \bmod (q-1)$ and $s' \equiv p^a r \bmod (q-1)$. In particular, $r' + s' \equiv r + s \bmod (p-1)$.

3.2 DECOMPOSITION OF DELIGNE–LUSZTIG CHARACTERS. In order to effectively use Corollary 3.1 and Jantzen's reduction formula, one has to decompose the Deligne–Lusztig characters $R_w(r,s) = R_w(\mu) = R_w(3,\mu)$ (in the notation of

[Jan 1]), $\mu = (r,s) = r\omega_1 + s\omega_2$, into irreducibles. Denote simple reflections of W by

$$s_1 : e_1 \leftrightarrow e_2, e_3 \mapsto e_3, \quad s_2 : e_2 \leftrightarrow e_3, e_1 \mapsto e_1.$$

Furthermore, set $w_0 : e_1 \leftrightarrow e_3, e_2 \mapsto e_2$, $d = \gcd(3, q-1)$.

First consider the characters $R_1(\mu)$ corresponding to the maximal torus

$$T_1^F = \{x = \mathrm{diag}(x_1, x_2, x_1^{-1}x_2^{-1}) | \; x_i \in \mathbb{F}_q^*\}$$

of G. Choose $\beta \in \mathbb{F}_{q^2}$ and $\gamma \in \mathbb{F}_{q^3}$ such that $\mathbb{F}_{q^2}^* = \langle\beta\rangle$, $|\gamma| = q^2+q+1$, and set $\alpha = \beta^{q+1}$. Furthermore, denote

$$\varepsilon = \exp\Big(\frac{2\pi i}{q-1}\Big), \quad \eta = \exp\Big(\frac{2\pi i}{q^2-1}\Big), \quad \tau = \exp\Big(\frac{2\pi i}{q^2+q+1}\Big).$$

If $\mu' = (r', s')$, then $R_1(\mu')$ is determined via the character $\theta(\mu')$ which acts on T_1^F as follows:

$$\theta(\mu')\colon \mathrm{diag}(\alpha^{k_1}, \alpha^{k_2}, \alpha^{-k_1-k_2}) \mapsto \varepsilon^{k_1(r'+s')+k_2 s'}.$$

For any $w \in W$, we write $\mu' \sim_w \mu''$ if the actions of $\theta(\mu')$ and $\theta(\mu'')$ on T_w^F are identical.

LEMMA 3.2: *Each $R_1(\mu)$ with μ regular is equal to exactly one $R_1(r,s)$ with (r,s) satisfying*

$$(6) \qquad 1 \le r, s, r+2s \le q-1, \quad 2r+s \le q-2.$$

Proof: Assume that $\mu = (r,s)$ is regular. One can suppose that $1 \le r, s \le q-2$ and $r+s \ne q-1$. If $r+s \ge q$, then for $w_0(\mu) = (-s,-r) \sim_1 (q-1-s, q-1-r)$ one has $(q-1-r)+(q-1-s) \le q-2$. We know that both μ and $w_0(\mu)$ are regular, and the scalar product $(R_1(\mu), R_1(w_0(\mu)))_G$ is at least 1. Therefore, $R_1(\mu) = R_1(w_0(\mu))$. Replacing μ by $w_0(\mu)$, one may suppose that $1 \le r, s, r+s \le q-2$. Suppose that $r+2s \ge q$. If $r \le s$, then for $s_1s_2(\mu) = (-r-s, r) \sim_1 (q-1-r-s, r)$ one has $(q-1-r-s)+r \le q-2$, $(q-1-r-s)+2r \le q-1$. If $r > s$, then for $s_2s_1(\mu) = (s, -r-s) \sim_1 (s, q-1-r-s)$ one has $s+(q-1-r-s) \le q-2$, $s+2(q-1-r-s) \le q-2$. Therefore, replacing μ by $s_1s_2(\mu)$ or $s_2s_1(\mu)$ if necessary, we can suppose that $\mu = (r,s)$ satisfies the condition $1 \le r, s, r+2s \le q-1$. Assume in addition now that $2r+s \ge q-1$. Then for $s_2s_1(\mu) = (s, -r-s) \sim_1 (s, q-1-r-s)$ one has $s+2(q-1-r-s) \le q-1$, $2s+(q-1-r-s) \le q-1$. In fact $2s+(q-1-r-s) \le q-2$, since the equality $2s+(q-1-r-s) = q-1$ would imply that $r = s = (q-1)/3$ and $s_2s_1(\mu) = \mu$, contrary to the regularity of

μ. Replacing now μ by $s_2s_1(\mu)$, we reach the effect that μ satisfies (6). Finally, direct calculations show that all $\mu = (r, s)$ with (r, s) satisfying (6) are regular and pairwise $W(T_1)^F$-inequivalent, as stated. ∎

Each $R_1(r, s)$ with (r, s) satisfying (6) is equal to exactly one of the characters $\chi^{(u,v,w)}_{(q+1)(q^2+q+1)}$ of degree $(q+1)(q^2+q+1)$. Recall that, in the notation of [SiF], $\mathrm{Irr}(G)$ consists of the following series of irreducible characters: 1_G, $\chi_{q(q+1)}$, $St = \chi_{q^3}$, $\chi^{(u)}_{q^2+q+1}$, $\chi^{(u)}_{q(q^2+q+1)}$, $\chi^{(u,v,w)}_{(q+1)(q^2+q+1)}$, $\chi^{(u)}_{q^3-1}$, $\chi^{(u)}_{(q-1)(q^2-1)}$, $\chi^{(u)}_{(q+1)(q^2+q+1)/3}$, $\chi^{(u)}_{(q-1)(q^2-1)/3}$ and $\chi^{(u)'}_{(q-1)(q^2-1)/3}$. Here, the last three series appear if and only if $q \equiv 1(\mathrm{mod}3)$, and the lower indices indicate the degree of the characters. Furthermore, it is not difficult to show that

$$R_1(0, s) = \chi^{(q-1-s)}_{q(q^2+q+1)} + \chi^{(q-1-s)}_{q^2+q+1}$$

if $1 \le s \le q-2$,

$$R_1\Big(\frac{q-1}{3}, \frac{q-1}{3}\Big) = \sum_{u=0}^{2} \chi^{(u)}_{(q+1)(q^2+q+1)/3}$$

(if $q \equiv 1(\mathrm{mod}3)$), and

$$R_1(0,0) = St + 2\chi_{q(q+1)} + 1_G.$$

The character $R_{s_1}(\mu)$, $\mu = (r, s)$, corresponds to the maximal torus $T^F_{s_1}$, which is conjugate to

$$\{x = x(k) = \mathrm{diag}(\beta^k, \beta^{kq}, \beta^{-k(q+1)}) \mid 0 \le k < q^2 - 1\},$$

and the character $\theta(\mu)$, which acts on $T^F_{s_1}$ as follows: $\theta(\mu) : x(k) \mapsto \eta^{k(r+s(q+1))}$. Arguing as in the proof of Lemma 3.2, we obtain

LEMMA 3.3: *Each $R_{s_1}(\mu)$ with μ regular is equal to exactly one $R_{s_1}(r, s)$ with (r, s) satisfying*

(7) $$1 \le r \le q-1, \; 0 \le s \le q-1-r.$$

Each $R_{s_1}(r, s)$ with (r, s) satisfying (7) is equal to exactly one of the characters $\chi^{(u)}_{q^3-1}$. Furthermore, it is not difficult to show that $R_{s_1}(0,0) = St - 1_G$, and

$$R_{s_1}(0, s) = \chi^{(q-1-s)}_{q(q^2+q+1)} - \chi^{(q-1-s)}_{q^2+q+1}$$

if $1 \le s \le q-2$.

The character $R_{s_1s_2}(\mu)$, $\mu = (r,s)$, corresponds to the maximal torus $T^F_{s_1s_2}$, which is conjugate to

$$\{y = y(k) = \mathrm{diag}(\gamma^k, \gamma^{kq}, \gamma^{kq^2}) |\, 0 \le k < q^2+q+1\},$$

and the character $\theta(\mu)$, which acts on $T^F_{s_1s_2}$ as follows: $\theta(\mu) : y(k) \mapsto \tau^{k(r+s(q+1))}$. Following the proof of Lemma 3.2, we obtain

LEMMA 3.4: *Each $R_{s_1s_2}(\mu)$ with μ regular is equal to exactly one $R_{s_1s_2}(r,s)$ with (r,s) satisfying one of the following three conditions:*

$$(8)\qquad \begin{array}{ll} \text{(a)} & r=0,\;\; 1\le s\le q-1,\\ \text{(b)} & 1\le r\le q,\;\; 0\le s\le q-1,\;\; 2r+s\le q,\;\; r+2s\le q,\\ \text{(c)} & 1\le r\le q,\;\; 0\le s\le q-2,\;\; 2r+s\ge 2q+3,\;\; r+2s\ge 2q. \end{array}$$

Each character $\chi^{(u)}_{(q-1)(q^2-1)}$ is equal to exactly one $R_{s_1s_2}(r,s)$ with (r,s) satisfying (8). Furthermore, it is not difficult to show that

$$R_{s_1s_2}(0,0) = St - \chi_{q(q+1)} + 1_G,$$

and (if $q \equiv 1(\mathrm{mod} 3)$)

$$\left\{R_{s_1s_2}\Big(\frac{q+2}{3},\frac{q-1}{3}\Big), R_{s_2s_1}\Big(\frac{q-1}{3},\frac{q+2}{3}\Big)\right\} = \left\{\sum_{u=0}^{2}\chi^{(u)}_{(q-1)(q^2-1)/3}, \sum_{u=0}^{2}\chi^{(u)'}_{(q-1)(q^2-1)/3}\right\}.$$

3.3 REDUCTION MODULO p. Recall we are assuming that a certain irreducible complex character χ of G leads to a GIR of $H \rhd G$. It is well known that $St \bmod p$ (and of course $1_G \bmod p$) belongs to $\mathrm{IBr}_p(G)$. In this subsection, using reduction modulo p (the defining characteristic), we show that there are only a few possibilities for χ. A crucial ingredient of our arguments is Jantzen's formula for reduction modulo p of Deligne–Lusztig characters (cf. [Jan 1], [Jan 2]), which states in the case $G = \mathrm{SL}_3(q)$ that

$$(9)\qquad R_w(\mu) \bmod p = \sum_{u\in W} \zeta(u(\mu - w\epsilon_{w_0u}) + q\rho_u - \rho),$$

where

$$\rho_u = \sum_{\alpha\in S, u^{-1}(\alpha)<0} \omega_\alpha, \quad \epsilon_u = u^{-1}\rho_u.$$

In what follows we shall write $R_w(\mu) \bmod p$ as a sum $\sum_{i=1}^{6}\zeta(\lambda^i)$ and $\lambda^i = (r^i, s^i)$. In general, (9) expresses $R_w(\mu) \bmod p$ as a linear combination of characters of

Weyl modules, some of which may occur with *negative* coefficient. Using this information and the results of §3.2, for most of $\psi \in \mathrm{Irr}(G)$ we single out certain irreducible constituents of $\psi \bmod p$ and then apply Corollary 3.1 to these constituents to show that ψ cannot be a candidate for our χ.

LEMMA 3.5: $\chi \neq \chi^{(u,v,w)}_{(q+1)(q^2+q+1)}$.

Proof: Assume the contrary: $\chi = \chi^{(u,v,w)}_{(q+1)(q^2+q+1)}$ for some u, v, w. Then, by Lemma 3.2, χ is equal to $R_1(r,s)$ with some (r,s) satisfying (6). Due to (9), $\chi \bmod p = \sum_{i=1}^{6} \zeta(\lambda^i)$, where all the weights $\lambda^1 = (r,s)$, $\lambda^2 = (q-r-1, r+s)$, $\lambda^3 = (r+s, q-s-1)$, $\lambda^4 = (s, q-r-s-1)$, $\lambda^5 = (q-r-s-1, r)$, $\lambda^6 = (q-s-1, q-r-1)$ belong to $X_f(T)$.

1) If $\chi \bmod p = \sum_{m=1}^{n} \varphi_m$ with $\varphi_m \in \mathrm{IBr}_p(G)$, then, by our hypothesis, $D = \deg\varphi_1 = \ldots = \deg\varphi_n$ divides the integers $\deg\chi = (q+1)(q^2+q+1)$ and $\deg\zeta(\lambda^i)$, $i = 1, \ldots, 6$. This condition implies that

$$\text{(10)}\qquad \begin{array}{c}\text{if } t \text{ is a prime divisor of } D, \text{ then}\\ \text{either } t = 2, \text{ or } t = 3|(q-1), \text{ or } t|(q+1).\end{array}$$

In particular,

$$\text{(11)}\qquad D \text{ divides } 3(q+1).$$

2) In what follows we shall use the p-adic decompositions of r, s, $r+s$:

$$r = \sum_{j=0}^{f-1} p^j r_j, \quad s = \sum_{j=0}^{f-1} p^j s_j, \quad r+s = \sum_{j=0}^{f-1} p^j t_j$$

$(0 \leq r_j, s_j, t_j \leq p-1)$. Also, we put

$$r'_j = p-1-r_j, \quad s'_j = p-1-s_j, \quad t'_j = p-1-t_j.$$

Observe that $p \neq 2$. Indeed, if $p = 2$, then $\deg\chi$ and D are odd. Hence, $(r_j, s_j) \neq (0,0), (1,1)$, which implies that $t_j = r_j + s_j = 1$. In this case, either $\deg\zeta(\lambda^2)$ or $\deg\zeta(\lambda^3)$ is divisible by $\deg\varphi(1,1) = 8$, a contradiction.

3) Next we show that

$$\text{(12)}\qquad 1 \leq r_j, s_j, t_j \leq p-2.$$

Indeed, if $r_j = 0$, then $r'_j = p-1$, which implies that $\deg\varphi(\lambda^6)$ is divisible by

$$\deg\varphi(s'_j, p-1)^{(p^j)} = \frac{p(s'_j+1)(p+s'_j+1)}{2},$$

i.e. $p|D$, contrary to (10). If $r_j = p-1$, then $\deg\varphi(\lambda^1)$ is divisible by p. Other possibilities $s_j, t_j \in \{0, p-1\}$ are excluded in a similar way.

4) Here we observe that $p \geq 5$ and $f \geq 2$. Indeed, if $p = 3$, then from (12) it follows that $r_j = s_j = 1$ for all j. But in this case one has $t_j = 2 = p-1$, contrary to (12). Suppose that $q = p \geq 5$. Then from (6) it follows that $1 \leq r, s, r+s \leq p-3$. By Corollary 3.1 we have

$$r+s \equiv s \equiv r \equiv -r \equiv -s \equiv -r-s \bmod (p-1),$$

i.e. $r \equiv s \equiv 0 \bmod (p-1)$, a contradiction.

5) This step is to eliminate the cases $p = 5, 7, 11, 13$. First suppose $p = 5$. By (12) and (11), $1 \leq r_j, s_j \leq 3$ and $D = \prod_{j=0}^{f-1} \deg\varphi(r_j, s_j)$ divides $3(5^f+1)$. The equality $\deg\varphi(1,1) = 8$ forces $(r_j, s_j) \neq (1,1)$. Hence, $\deg\varphi(r_j, s_j) \geq 15$ and so $D \geq 15^f > 3(5^f+1)$, a contradiction. Second, suppose $p = 7$. Then $1 \leq r_j, s_j \leq 5$ and D divides $3(7^f+1)$. If f is even, then $\deg\varphi(1,1) = 8$ does not divide $3(7^f+1)$. Hence, $(r_j, s_j) \neq (1,1)$ for all j, which implies that $D \geq 15^f > 3(7^f+1)$, a contradiction. If f is odd, then $3(7^f+1)$ is not divisible by 16. Therefore, (r_j, s_j) can be equal to $(1,1)$ for at most one index j. This forces $D \geq 8 \cdot 15^{f-1} > 3(7^f+1)$, again a contradiction. Next, assume $p = 11$. Then $1 \leq r_j, s_j \leq 9$ and D divides $3(11^f+1)$. Since $3(11^f+1)$ is not divisible by 5 and by 8, $(r_j, s_j) \neq (1,1), (1,2), (2,1)$. Hence $\deg\varphi(r_j, s_j) \geq 24$, which implies that $D \geq 24^f > 3(11^f+1)$, a contradiction. Lastly suppose $p = 13$. Then $1 \leq r_j, s_j \leq 11$ and D divides $3(13^f+1)$. Since $3(13^f+1)$ is not divisible by 8 and by 9, $(r_j, s_j) \neq (1,1), (2,2), (1,3), (3,1)$. Furthermore, (r_j, s_j) can belong to $\{(1,2), (2,1)\}$ for at most one index $j = j_0$. For $j \neq j_0$ one has $\deg\varphi(r_j, s_j) \geq 35$. Hence, $D \geq 15 \cdot 35^{f-1} > 3(13^f+1)$, a contradiction.

6) As a result of the previous steps, we may suppose that $f \geq 2$ and $p \geq 17$. Observe in addition to (12) that $r_j + s_j \notin \{p-3, p-2, p-1, p, p+1\}$. Indeed, if $r_j + s_j = p-2$, then $\deg\varphi(r_j, s_j)$ is divisible by p (see (5) and Lemma 2.5), and so $p|D = \deg\varphi(\lambda^1)$, contrary to (10). If $r_j + s_j = p$, then $r'_j + s'_j = p-2$, which implies that p divides $D = \deg\varphi(\lambda^6)$, a contradiction. If $r_j + s_j = p-3$, then $D = \deg\varphi(\lambda^1)$ is divisible by $(p-1)/2$. By (11) one has $(p-1)/2$ divides $3(p^f+1)$, i.e. $(p-1)/2$ divides 6, which is impossible as $p \geq 17$. If $r_j + s_j = p+1$, then $r'_j + s'_j = p-3$, and we come again to the contradiction that $(p-1)|12$. Finally, assuming $r_j + s_j = p-1$ we have $t_j \in \{p-1, 0\}$, contrary to (12).

In connection with this observation, set $J = \{j|\ r_j + s_j \geq p+2\}$, $J' = \{j|\ r'_j + s'_j \geq p+2\}$. It is clear that $J' = \{0, 1, \ldots, f-1\} \backslash J$. Therefore, without loss of generality one can suppose that $|J| \geq f/2$. From (5) and Lemma 2.5 it follows that $\deg\varphi(r_j, s_j) > 2p^2$ whenever $j \in J$. Consequently, $D = \deg\varphi(\lambda^1) =$

$\prod_{j=0}^{f-1} \deg \varphi(r_j, s_j) \geq 2^{|J|}q$. In particular, if $f \geq 3$ or if $|J| \geq 2$ one has $D \geq 4q$, contrary to (11). If $f = 2$ and $|J| = 1$, then $D \geq 8 \cdot 2p^2 = 16q > 3(q+1)$, again a contradiction. This contradiction completes the proof of Lemma 3.5. ∎

LEMMA 3.6: $\chi \neq \chi_{q^3-1}^{(u)}$.

Proof: Assume the contrary: $\chi = \chi_{q^3-1}^{(u)}$ for some u. Then, by Lemma 3.3, χ is equal to $R_{s_1}(r,s)$ with some (r,s) satisfying (7). By (9), $\chi \bmod p = \sum_{i=1}^{6} \zeta(\lambda^i)$, where $\lambda^1 = (r-2, s+1)$, $\lambda^2 = (q-r-1, r+s)$, $\lambda^3 = (r+s-1, q-s-2)$, $\lambda^4 = (s-1, q-r-s-2)$, $\lambda^5 = (q-r-s, r-2)$, $\lambda^6 = (q-s-1, q-r-1)$. It is obvious that $\lambda^2, \lambda^3, \lambda^6 \in X_f(T)$; furthermore, if $i = 1,4,5$, then either λ^i belongs to $X_f(T)$, or $\zeta(\lambda^i) = 0$. In particular, $\chi \bmod p$ contains $\varphi(\lambda^i)$ with $i = 2,3,6$. By Corollary 3.1, $s \equiv r-2 \equiv -r-s \bmod (p-1)$, i.e. $s \equiv r-2 \bmod (p-1)$ and $3r \equiv 4 \bmod (p-1)$.

First suppose that $r \geq 2$. Then $\chi \bmod p$ also contains $\varphi(\lambda^1)$, and so by Corollary 3.1 one has $r+s-1 \equiv s \bmod (p-1)$. This implies that $r \equiv 1 \bmod (p-1)$ and so $p = 2$. Observe that there exists $i_0 \in \{1,2,3,6\}$ such that $r^{i_0} \equiv s^{i_0} \equiv 1 \pmod 2$. From this it follows that $\deg \varphi(\lambda^{i_0})$ is divisible by $\deg \varphi(1,1) = 8$. In the meantime, we know that $\deg \varphi(\lambda^{i_0})$ divides $\deg \chi = q^3 - 1$ and so it is an odd integer, a contradiction.

It now remains to consider the case $r = 1$. In this case we again have $3 \equiv 4 \bmod (p-1)$, whence $p = 2$. If $1 \leq s \leq q-3$, then $\chi \bmod p$ contains $\varphi(\lambda^i)$ with $i = 3,4$. This assumption leads as above to a contradiction, because there exists $i_1 \in \{3,4\}$ such that $r^{i_1} \equiv s^{i_1} \equiv 1 \pmod 2$. Finally, if $r = 1$ and $s \in \{0, q-2\}$, then $\chi \bmod p$ contains some $\varphi(\lambda^{i_2})$ with $\{r^{i_2}, s^{i_2}\} = \{q-2, q-1\}$. We again arrive at a contradiction, because $\deg \varphi(\lambda^{i_2}) = 3 \cdot 8^{f-1}$ does not divide $\deg \chi = 8^f - 1$.
∎

LEMMA 3.7: *If* $\chi = \chi_{(q-1)(q^2-1)}^{(u)}$ *for some* u, *then* $q = 2$.

Proof: Assume the contrary: $\chi = \chi_{(q-1)(q^2-1)}^{(u)}$ for some u. Then, by Lemma 3.4, χ is equal to $R_{s_1 s_2}(r,s)$ with some (r,s) satisfying (8).

1) First we consider the case $\chi = R_{s_1 s_2}(r', s')$ with some (r', s') satisfying (8) (c). Then $\chi = R_{s_2\, s_1}(r,s)$ with $1 \leq r \leq q-4$, $2 \leq s \leq q-2$, $2r+s \leq q$, $r+2s \leq q$, by (9), $\chi \bmod p = \sum_{i=1}^{6} \zeta(\lambda^i)$, where $\lambda^1 = (r, s-3)$, $\lambda^2 = (q-r-2, r+s-1)$, $\lambda^3 = (r+s-2, q-s)$, $\lambda^4 = (s-1, q-r-s-2)$, $\lambda^5 = (q-r-s, r-2)$, $\lambda^6 = (q-s-1, q-r-1)$. It is obvious that λ^2, λ^3, λ^4, $\lambda^6 \in X_f(T)$; furthermore, if $i = 1,5$, then either λ^i belongs to $X_f(T)$, or $\zeta(\lambda^i) = 0$. In particular, $\chi \bmod p$

contains $\varphi(\lambda^i)$ with $i = 2,3,4,6$. By Corollary 3.1, $s-2 \equiv r-1 \equiv -r-2 \equiv -r-s \bmod (p-1)$, i.e. $s \equiv r+1 \equiv 2 \bmod (p-1)$ and $(p-1)|3$. Hence, $p=2$. Observe that there exists $i_0 \in \{2,3,4,6\}$ such that $r^{i_0} \equiv s^{i_0} \equiv 1(\text{mod}2)$. From this it follows that $\deg\varphi(\lambda^{i_0})$ is divisible by $\deg\varphi(1,1) = 8$. In the meantime, we know that $\deg\varphi(\lambda^{i_0})$ divides $\deg\chi = (q-1)(q^2-1)$ and so it is an odd integer, a contradiction.

2) In the rest of the proof we can suppose that $\chi = R_{s_1s_2}(r,s)$ with some (r,s) satisfying (8)(a) or (8)(b). Then by (9), $\chi \bmod p = \sum_{i=1}^{6}\zeta(\lambda^i)$, where $\lambda^1 = (r-3,s)$, $\lambda^2 = (q-r, r+s-2)$, $\lambda^3 = (r+s-1, q-s-2)$, $\lambda^4 = (s-2, q-r-s)$, $\lambda^5 = (q-r-s-2, r-1)$, $\lambda^6 = (q-s-1, q-r-1)$. By Lemma 2.3, either $\lambda^i \in X_f(T)$, or $\zeta(\lambda^i) = 0$, or else $i = 1,4$ and the following holds:

$$i=1,\quad \zeta(\lambda^1) = \begin{cases} \zeta(0,0), & (r,s) = (0,0), \\ -\zeta(1-r, r+s-2), & 0 \le r \le 1, s \ge 2-r; \end{cases}$$

$$i=4,\quad \zeta(\lambda^4) = \begin{cases} -\zeta(0, q-1-r), & s = 0, \\ 0, & s = 1. \end{cases}$$

In particular, if $s = 0$, then by Corollary 2.2, $\chi \bmod p$ contains $\varphi(\lambda^6) = \varphi(q-1, q-r-1)$ with $1 \le r \le q/2$. If $p > 2$, then $\deg\varphi(\lambda^6)$ is divisible by p, which is impossible because $p \nmid \deg\chi$. If $p = 2$ and $r < q-1$, then $\deg\varphi(\lambda^6)$ is divisible by 8, again a contradiction. Therefore $q = 2$ and $r = 1$. In what follows we may suppose that $s \ge 1$.

3) Here we consider the case $r \ge 3$. Then $\chi \bmod p$ contains $\varphi(\lambda^i)$ with $i = 1,2,3,6$. By Corollary 3.1 one has $r+s-3 \equiv s-1 \equiv r-2 \equiv -r-s \bmod (p-1)$. This implies that $r \equiv s+1 \equiv 2 \bmod (p-1)$ and $(p-1)|3$. In particular, $p=2$. In this case, there exists $i_1 \in \{1,2,3,6\}$ such that $r^{i_1} \equiv s^{i_1} \equiv 1 \pmod 2$. From this it follows that $\deg\varphi(\lambda^{i_1})$ is divisible by $\deg\varphi(1,1) = 8$. In the meantime, we know that $\deg\varphi(\lambda^{i_1})$ divides $\deg\chi = (q-1)(q^2-1)$, a contradiction.

4) Next suppose that $r = 2$. Then $q \ge 5$, and $\chi \bmod p$ contains $\varphi(\lambda^i)$ with $i = 2,3,6$. By Corollary 3.1, $s-1 \equiv 0 \equiv -s-2 \bmod (p-1)$, i.e. $p = 2$. Now if s is odd, then $\deg\varphi(\lambda^6) = \deg\varphi(q-s-1, q-3)$ is divisible by 8, a contradiction. If s is even, then $s \ge 2$ and $\deg\varphi(\lambda^2) = \deg\varphi(q-2, s)$ is divisible by 8, again a contradiction.

5) If $r = 1$, then by Corollary 2.2, $\chi \bmod p$ contains $\varphi(\lambda^2) = \varphi(q-1, s-1)$ and $\varphi(\lambda^6) = \varphi(q-s-1, q-2)$. If $p > 2$, or if $p = 2$ and $s \ge 2$, then $\deg\varphi(\lambda^2)$ is divisible by p, a contradiction. Therefore $p = 2$ and $s = 1$. In this case $q = 2^f \ge 4$, hence $\deg\varphi(\lambda^6)$ is divisible by 8, again a contradiction.

6) Finally, suppose that $r = 0$. Then $1 \le s \le q-1$. Again by Corollary

2.2 $\chi \bmod p$ contains $\varphi(\lambda^6) = \varphi(q-s-1, q-1)$. If $p > 2$, or if $p = 2$ and $s \leq q-2$, then $\deg \varphi(\lambda^6)$ is divisible by p, a contradiction. Therefore $p = 2$ and $s = q-1$. If $q \geq 8$, then $\chi \bmod p$ contains $\varphi(\lambda^4) = \varphi(q-3, 1)$, whose degree is divisible by 8, again a contradiction. If $q = 4$, then $\chi \bmod p = \zeta(4,1) + \zeta(0,3) = \zeta(4,1) + \varphi(0,3) + \varphi(0,0)$, contrary to our hypothesis on χ. Hence, $q = 2$ and $s = 1$.

Lemma 3.7 is completely proved. ∎

Lemma 3.8: $\chi \neq \chi^{(u)}_{q(q^2+q+1)}, \chi^{(u)}_{q^2+q+1}$.

Proof: First assume that $\chi = \chi^{(u)}_{q(q^2+q+1)}$ for some u. Then, by the results of §3.2 and (9),

$$\begin{aligned} \chi \bmod p &= \tfrac{1}{2}(R_1(0,s) + R_{s_1}(0,s)) \bmod p \\ &= \zeta(q-1,s) + \zeta(s-1,q-s-2) + \zeta(s,q-s-1) + \zeta(q-s-1,q-1) \end{aligned}$$

for some s, $1 \leq s \leq q-2$. In particular, $\chi \bmod p$ contains $\varphi(q-1,s)$ and $\varphi(s,q-s-1)$. Writing down $s = \sum_{j=0}^{f-1} p^j s_j$, $0 \leq s_j \leq p-1$, we have

$$\deg \varphi(q-1,s) = \prod_{j=0}^{f-1} \deg \varphi(p-1,s_j) = \prod_{j=0}^{f-1} \frac{p(s_j+1)(p+s_j+1)}{2},$$

$$\deg \varphi(s,q-s-1) = \prod_{j=0}^{f-1} \deg \varphi(s_j,p-s_j-1) = \prod_{j=0}^{f-1} \frac{p(p-1)+2(s_j+1)(p-s_j)}{2}.$$

Since $s \geq 1$, from these formulas it follows that $\deg \varphi(s,q-s-1) < \deg \varphi(q-1,s)$, a contradiction.

Now assume that $\chi = \chi^{(u)}_{q^2+q+1}$ for some u. Then χ is an irreducible Weil character of G, hence our statement follows from §3 of [Tiep 3]. ∎

Lemma 3.9: *If $\chi = \chi_{q(q+1)}$, then $q = p$.*

Proof: Assume that $\chi = \chi_{q(q+1)}$. Then, by the results of §3.2 and (9),

$$\chi \bmod p = \left(\frac{R_1(0,0) - R_{s_1}(0,0)}{2} - 1_G \right) \bmod p = \zeta(0,q-1) + \zeta(q-1,0).$$

In particular, $\chi \bmod p$ contains $\varphi(0,q-1)$ and $\varphi(q-1,0)$, both of degree $(p(p+1)/2)^f$. By our hypothesis, the last integer must divide $\deg \chi = p^f(p^f+1)$. This condition immediately implies that $f = 1$, as stated. ∎

Observe that the three characters $\chi^{(u)}_{(q+1)(q^2+q+1)/3}$, $0 \le u \le 2$, are $\tilde{G}$-conjugate, where $\tilde{G} = GL_3(q)$, and the six characters

$$\chi^{(u)}_{(q-1)(q^2-1)/3}, \quad \chi^{(u)'}_{(q-1)(q^2-1)/3}, \quad 0 \le u \le 2,$$

are $\mathrm{Aut}(G)$-conjugate. This can easily be seen by inspecting the character table of G (cf. [SiF]). As a consequence of this observation, we come to the following statement.

COROLLARY 3.10: *Suppose that* $\chi = \chi^{(u)}_{(q+1)(q^2+q+1)/3}$ (resp., $\chi = \chi^{(u)}_{(q-1)(q^2-1)/3}$ *or* $\chi^{(u)'}_{(q-1)(q^2-1)/3}$), *and that* $\varphi(\lambda)$, $\varphi(\lambda')$ *are arbitrary irreducible constituents of* $R_1(\frac{q-1}{3}, \frac{q-1}{3}) \bmod p$ *(of* $(R_{s_1 s_2}(\frac{q+2}{3}, \frac{q-1}{3}) + R_{s_2 s_1}(\frac{q-1}{3}, \frac{q+2}{3})) \bmod p$ *respectively). Then* λ *and* λ' *satisfy the conclusions of Corollary 3.1.*

LEMMA 3.11: *Suppose that* $q \equiv 1 \pmod 3$. *Then*

$$\chi \neq \chi^{(u)}_{(q+1)(q^2+q+1)/3}, \quad \chi^{(u)}_{(q-1)(q^2-1)/3}, \quad \chi^{(u)'}_{(q-1)(q^2-1)/3}.$$

Proof: Assume the contrary.

1) Suppose that $\deg \chi = (q-1)(q^2-1)/3$. Then we apply Corollary 3.10 to

$$\left(R_{s_1 s_2}(\tfrac{q+2}{3}, \tfrac{q-1}{3}) + R_{s_2 s_1}(\tfrac{q-1}{3}, \tfrac{q+2}{3})\right) \bmod p$$

$$= 3\left(\zeta(\tfrac{q-1}{3}, \tfrac{q-7}{3}) + \zeta(\tfrac{q-7}{3}, \tfrac{q-1}{3}) + \zeta(\tfrac{2q-2}{3}, \tfrac{2q-5}{3}) + \zeta(\tfrac{2q-5}{3}, \tfrac{2q-2}{3})\right).$$

If $q \ge 7$, then

$$\frac{q-1}{3} + \frac{q-7}{3} \equiv \frac{2q-2}{3} + \frac{2q-5}{3} \bmod (p-1)$$

by Corollary 3.10, i.e. $p = 2$. But in this case, $\deg \varphi(\frac{q-1}{3}, \frac{q-7}{3})$ is divisible by 8, a contradiction. If $q < 7$, then $q = 4$, and

$$(R_{s_1 s_2}(2,1) + R_{s_2 s_1}(1,2)) \bmod p = 3(\zeta(2,1) + \zeta(1,2)).$$

This character contains the constituent $\varphi(2,1)$, whose degree is equal to 9 and so does not divide $\deg \chi = 15$, again a contradiction.

2) Next we consider the case $\chi = \chi^{(u)}_{(q+1)(q^2+q+1)/3}$. By (9),

$$R_1\left(\frac{q-1}{3}, \frac{q-1}{3}\right) \bmod p = 3\left(\zeta\left(\frac{q-1}{3}, \frac{q-1}{3}\right) + \zeta\left(\frac{2q-2}{3}, \frac{2q-2}{3}\right)\right).$$

Clearly, $R_1(\frac{q-1}{3}, \frac{q-1}{3}) \bmod p$ contains $\varphi(\lambda^i)$, $i = 1, 2$, where

$$\lambda^1 = \left(\frac{q-1}{3}, \frac{q-1}{3}\right), \quad \lambda^2 = \left(\frac{2q-2}{3}, \frac{2q-2}{3}\right).$$

Our hypothesis on χ implies that $\deg\varphi(\lambda^1) = \deg\varphi(\lambda^2)$ and that this integer divides $(q+1)(q^2+q+1)$. First suppose $p \equiv 1 \pmod 3$. Then

$$\deg\varphi(\lambda^1) = \deg\varphi\left(\sum_{j=0}^{f-1} p^j\left(\frac{p-1}{3}, \frac{p-1}{3}\right)\right) = \left(\frac{p+2}{3}\right)^{3f},$$

$$\deg\varphi(\lambda^2) = \deg\varphi\left(\sum_{j=0}^{f-1} p^j\left(\frac{2p-2}{3}, \frac{2p-2}{3}\right)\right) = \left(\frac{(2p+1)^3}{27} - \frac{(p-1)^3}{27}\right)^f.$$

Therefore, we come to the equality $(p+2)^3 = (2p+1)^3 - (p-1)^3$, a contradiction. Finally, suppose that $p \equiv -1 \pmod 3$. Since $q \equiv 1 \pmod 3$, one has $f = 2g$ for some $g \in \mathbb{N}$. Then

$$\deg\varphi(\lambda^1) = \left(\deg\varphi\left(\frac{2p-1}{3}, \frac{2p-1}{3}\right) \cdot \deg\varphi\left(\frac{p-2}{3}, \frac{p-2}{3}\right)\right)^g.$$

Since

$$\deg\varphi\left(\frac{p-2}{3}, \frac{p-2}{3}\right) = \left(\frac{p+1}{3}\right)^3$$

and

$$\deg\varphi\left(\frac{2p-1}{3}, \frac{2p-1}{3}\right) = \left(\frac{2p+2}{3}\right)^3 - \left(\frac{p-2}{3}\right)^3,$$

$\deg\varphi(\lambda^1)$ is divisible by 2^{3g}, and so it does not divide $(q+1)(q^2+q+1)$, again a contradiction. ∎

Corollary 3.12: *Let $\chi \in \mathrm{Irr}(G)$, where $G = \mathrm{SL}_3(q)$ and $q = p^f$. Then $\chi \bmod p$ is absolutely irreducible if and only if either $\chi \in \{1_G, St\}$, or $q = 2$ and $\chi \in \{\chi^{(u)}_{(q-1)(q^2-1)} \mid u = 1, 2\}$.*

Observe that $\chi_{q(q+1)} \bmod p$ is *not* irreducible if $f = 1$: it is a sum of two $\mathrm{Aut}(G)$-conjugate Brauer characters, see the proof of Lemma 3.9.

3.4 Proof of Theorem 1.3. Assume that G is a covering group of $L = L_3(q)$, $q = p^f$, and that $\chi \in \mathrm{Irr}(G)$ leads to a faithful GIR of H, $H \rhd G$.

If G is a factor group of $\mathrm{SL}_3(q)$, then without loss of generality one can suppose that $G = \mathrm{SL}_3(q)$. By the results of §3.3, either $q = 2$ and $\chi = \chi^{(u)}_{(q-1)(q^2-1)}$ for some u, or $\chi \in \{St, \chi_{q(q+1)}\}$. (The principal character 1_G is excluded because χ is faithful.) The former possibility can in fact be realized, since $\mathrm{SL}_3(2) \simeq L_2(7)$; and here we arrive at conclusion (i) of Theorem 1.3. Consider the latter case. It is well known that St and $\chi_{q(q+1)}$ are extendible to the whole group $\tilde{G} = \mathrm{GL}_3(q)$.

Let χ_1, resp. χ_2, be an extension of St, resp. of $\chi_{q(q+1)}$, to $\tilde{G}$. Choose a prime divisor t of q^2+q+1. Then by [Jam] there exist $\varphi_1, \varphi_2 \in \mathrm{IBr}_t(\tilde{G})$ such that

$$\chi_1 \bmod t = \varphi_1 + \varphi_2, \quad \chi_2 \bmod t = 1 + \varphi_1.$$

Thus $\deg \varphi_1 = q^2+q-1$, $\deg \varphi_2 = (q-1)(q^2-1)$, and so by Proposition 1.2 we get a contradiction.

The case $G = 2 \cdot \mathrm{SL}_3(2) \simeq \mathrm{SL}_2(7)$ has been considered in [Tiep 1].

To complete the proof of Theorem 1.3, it remains to consider the case where $L = L_3(4)$, G is not a factor group of $\mathrm{SL}_3(4)$. Here, $\mathrm{Mult}(L) = \mathbb{Z}_4 \times \mathbb{Z}_{12}$, therefore we have to look at the faithful irreducible characters of G, G is one of the groups $2L$, 4_1L, 4_2L, $6L$, 12_1L, 12_2L (the notation is taken from [ATLAS]). The irreducible characters of these groups are given in [ATLAS], and the Brauer characters in [JLPW]. Using this information and Proposition 1.2, one can verify that (G,χ) satisfies one of the following conditions.

1) $G = 4_1L$ and $\deg\chi = 8$. G has exactly 4 faithful irreducible characters α_i, $i = 1,\dots,4$, of degree 8. Here $\alpha_2 = \bar{\alpha_1}$, $\alpha_4 = \bar{\alpha_3}$, $\mathbb{Q}(\alpha_i) = \mathbb{Q}(\sqrt{-1},\sqrt{5})$, and $\alpha_1,\dots,\alpha_4$ are algebraically conjugate. Set $G_1 = G \cdot 2_1$, $\tilde{\alpha} = \mathrm{Ind}_G^{G_1}(\alpha_1)$. Then $\tilde{\alpha} \in \mathrm{Irr}(G_1)$, $\tilde{\alpha}|_G = \alpha_1+\alpha_4$, and $\mathbb{Q}(\tilde{\alpha}) = \mathbb{Q}(\sqrt{-5})$. Moreover, all prime reductions of $\tilde{\alpha}$ are absolutely irreducible. Therefore, there exists a GIR of G_1 of dimension 32, which affords the G_1-character $\tilde{\alpha} + \bar{\tilde{\alpha}}$ (and the G-character $\alpha_1 + \cdots + \alpha_4$).

2) $G = 6L$ and $\deg\chi = 6$. G has exactly 2 faithful irreducible characters β_i, $i = 1,2$, of degree 6. Here $\beta_2 = \bar{\beta}_1$ and $\mathbb{Q}(\beta_i) = \mathbb{Q}(\sqrt{-3})$. Set $G_2 = G \cdot 2_2$, $\tilde{\beta} = \mathrm{Ind}_G^{G_2}(\beta_1)$. Then $\tilde{\beta} \in \mathrm{Irr}(G_2)$, $\tilde{\beta}|_G = \beta_1+\beta_2$, $\mathbb{Q}(\tilde{\alpha}) = \mathbb{Q}$, and $\mathrm{ind}(\tilde{\beta}) = -1$. Moreover, the prime reductions $\tilde{\beta} \bmod t$ are absolutely irreducible (and are of type $-$) if $t \geq 5$; if $t \leq 3$ then they are sums of two absolutely irreducible Brauer characters, which cannot be written over the prime field $\mathbb{F}_t$. Therefore, there exists a GIR V of G_2 of dimension 24, which affords the G_2-character $2\tilde{\beta}$ (and the G-character $2(\beta_1+\beta_2)$). Observe that $G_2 \hookrightarrow 6Suz \cdot 2 \hookrightarrow 2Co_1$, and the corresponding lattices are isometric to the Leech lattice.

3) $G = 12_1L$ and $\deg\chi = 24$. G has exactly 8 faithful irreducible characters γ_i, $i = 1,\dots,8$, of degree 24. Here

$$\mathbb{Q}(\gamma_i) = \mathbb{Q}\left(\exp\frac{\pi i}{6}, \sqrt{-7}\right),$$

and $\gamma_1,\dots,\gamma_8$ are algebraically conjugate. All prime reductions $\gamma_i \bmod t$ are absolutely irreducible. We do not know examples of GIR's of $H \rhd G$, which afford G-character $\kappa(\gamma_1+\cdots+\gamma_8)$ for some $\kappa \in \mathbb{N}$.

Theorem 1.3 is completely proved. ∎

4. The groups $\mathrm{SU}_3(q)$

4.1 DECOMPOSITION OF DELIGNE–LUSZTIG CHARACTERS. We keep the notation of §2. The twisted group G of Lie type ${}^2A_2$ over $\mathbb{F}_q$, $q = p^f$, is defined by means of the automorphism $\pi : \alpha_1 \leftrightarrow \alpha_2$ of the Dynkin diagram of type A_2, and Fr^f, where Fr denotes the Frobenius endomorphism $Fr : X = (x_{ij}) \mapsto X^{(p)} = (x_{ij}^p)$ of $\mathcal{G}$. First, one extends π to an involutive automorphism of $\mathcal{G}$ by setting $\pi(X) = A \cdot {}^tX^{-1} \cdot A^{-1}$ for $X \in \mathcal{G}$, where

$$A = \begin{pmatrix} & & 1 \\ & 1 & \\ 1 & & \end{pmatrix}.$$

In particular, $\pi(X) = \mathrm{diag}(c^{-1}, b^{-1}, a^{-1})$ for any $X = \mathrm{diag}(a,b,c) \in T$. Then π acts on $X(T)$ via $\pi(\lambda) = \lambda \circ (\pi^{-1}|_T)$ for any $\lambda \in X(T)$. Set $F = \pi \circ Fr^f$. Clearly, π and Fr^f commute with each other, and $F^2 = Fr^{2f}$. Now G is defined to be $\mathcal{G}^F = \{X \in \mathcal{G} |\ F(X) = X\}$. Thus G is the group of all matrices $X \in \mathrm{SL}_3(q^2)$ which preserve the Hermitian form with matrix A, i.e. $G \cong \mathrm{SU}_3(q)$. Since T is contained in the F-stable Borel subgroup

$$B = \left\{ \begin{pmatrix} * & * & * \\ 0 & * & * \\ 0 & 0 & * \end{pmatrix} \right\},$$

T is a maximally split torus. The maximal tori in G are of the form T_w^F, where $w \in W$, and it is known that $|T_w^F| = |\det(wF-1)|_{X(T)}|$ (cf. [Car]). In particular, $|T_1^F| = q^2 - 1$, $|T_{w_0}^F| = (q+1)^2$, where $w_0 = -\pi$ is the reflection corresponding to the root $\alpha_0 = e_1 - e_3$, and $|T_{s_1}^F| = q^2 - q + 1$, where s_1 is the reflection corresponding to the root α_1. Furthermore, G has three conjugacy classes of maximal tori, with representatives T_1^F, $T_{w_0}^F$, and $T_{s_1}^F$. Correspondingly, it suffices to consider the Deligne–Lusztig characters $R_1(\mu)$, $R_{w_0}(\mu)$, and $R_{s_1}(\mu)$.

LEMMA 4.1: *For $w \in W$, $W(T_w)^F := N_{\mathcal{G}}(T_w)^F / T_w^F$ is equal to*

$$\{u \in W |\ [u, ww_0] = 1\}.$$

Proof: For any $u \in W$, let the action of u on T be induced by the permutation matrix n_u. Then it is clear that $n_{uv} = n_u n_v$, ${}^tn_u = n_{u^{-1}}$, $(n_u)^{(q)} = n_u$, and $A = n_{w_0}$. By definition, $T_w = g_w T g_w^{-1}$, where $g_w \in \mathcal{G}$ is such that $g_w^{-1} \cdot F(g_w) = n_w$. Now let $g \in N_{\mathcal{G}}(T_w)^F$. Then $g = g_w \cdot n_u t \cdot g_w^{-1}$ for some $t \in T$ and $u \in W$. But $F(g) = g$, hence we get $ts^{-1} = n_{u^{-1}ww_0uw_0^{-1}w^{-1}}$, where $s =$

$n^{-1}_{w_0^{-1}w^{-1}}(t^{-1})^{(q)}n_{w_0^{-1}w^{-1}} \in T$. Thus $s = t$, and $[u, ww_0] = 1$. Observe that the equality $s = t$ is equivalent to $t = n_w F(t) n_w^{-1}$, i.e. $g_w t g_w^{-1} \in T_w^F$. Conversely, if $u \in W$ is such that $[u, ww_0] = 1$, then setting $g = g_w n_u g_w^{-1}$ we get $g \in N_{\mathcal{G}}(T_w)^F$. ∎

Now we want to decompose the Deligne–Lusztig characters $R_w(r,s) = R_w(\mu) = R_w(3,\mu)$ (in the notation of [Jan 1]), $\mu = (r,s) = r\omega_1 + s\omega_2$, into irreducibles. We shall make use of the following identities (cf. [Jan 1]):

$$R_w(\mu) = R_{uww_0u^{-1}w_0}(u\mu) \tag{13}$$

for any $u \in W$, and

$$R_w(\mu) = R_w(\mu + (q - w\pi)\nu) \tag{14}$$

for any $\nu \in X(T)$.

First consider the characters $R_1(r,s) = R_1(\mu)$ corresponding to the maximal torus

$$T_1^F = \{\mathrm{diag}(a, a^{q-1}, a^{-q}) |\ a \in \mathbb{F}_{q^2}^*\}$$

of G, $\mu = (r,s) \in X(T)$.

LEMMA 4.2: *Each $R_1(\mu)$ is equal to exactly one $R_1(a,b)$ with (a,b) satisfying*

$$0 \le a \le q-2-b, \quad 0 \le b \le q-2, \quad (a,b) \ne (0,0) \tag{15}$$

if μ is regular, and

$$(a,b) = (0,0), \quad (c, q-1-c), \quad 0 \le c \le q-1 \tag{16}$$

if μ is not regular.

Proof: (14) implies that $R_1(r,s) = R_1(r',s')$ whenever $r + qs \equiv r' + qs' \bmod(q^2-1)$ (in this case we write $(r,s) \sim_1 (r',s')$ for brevity), and (13) implies that $R_1(\mu) = R_1(w_0\mu)$. Writing $r + s = t_0 + (q-1)x$, $s + x = s_0 + (q+1)y$ with $x, y, s_0, t_0 \in \mathbb{Z}$, $0 \le s_0 \le q$, $0 \le t_0 \le q-2$, one sees that $(r,s) \sim_1 (r_0, s_0)$ with $r_0 = t_0 - s_0$, whence $R_1(r,s) = R_1(r_0,s_0)$. Thus one can suppose that $\mu = (r,s)$ with $0 \le s \le q$ and $0 \le r + s \le q-2$. If $r \ge 0$ then (r,s) satisfies (15). Suppose $r \le -1$ and $r + s > 0$. Then $w_0(\mu) \sim_1 (r',s')$ with $r' = q - s$, $s' = -r-1$; hence $R_1(r,s) = R_1(r',s')$ and (r',s') satisfies (15). Similarly, if $r \le -1$ and $r + s = 0$, then $R_1(r,s) = R_1(r',s')$ for the same r', s', and (r',s') satisfies (16). Recall that here μ is regular if and only if $\mu \not\sim_1 w(\mu)$ for any nontrivial $w \in W(T_1)^F$. Due to Lemma 4.1, the regularity of $\mu = (r, s)$ is now equivalent to that $r + s \not\equiv 0 \bmod (q-1)$. Therefore our statement follows. ∎

Recall that, in the notation of [Geck], $\mathrm{Irr}(G)$ consists of the following series of irreducible characters:

$$1_G,\ \chi_{q(q-1)},\ St = \chi_{q^3},\ \chi^{(u)}_{q^2-q+1},\ \chi^{(u)}_{q(q^2-q+1)},\ \chi^{(u,v)}_{(q-1)(q^2-q+1)},$$
$$\chi^{(u)}_{q^3+1},\ \chi^{(u)}_{(q+1)(q^2-1)},\ \chi^{(u)}_{(q-1)(q^2-q+1)/3},\ \chi^{(u,1)}_{(q+1)(q^2-1)/3},\ \chi^{(u,2)}_{(q+1)(q^2-1)/3}.$$

Here, the last three series appear if and only if $q \equiv -1 \pmod 3$, and the lower indices indicate the degree of the characters. Now it is clear that each of $(q+1)(q-2)/2$ characters $\chi^{(u)}_{q^3+1}$ is equal to one of $R_1(a, b)$ with (a, b) satisfying (15). It is not difficult to show that $R_1(0, 0) = 1_G + St$ and

$$R_1(a, q-1-a) = \chi^{(m_a)}_{q(q^2-q+1)} + \chi^{(m_a)}_{q^2-q+1},$$

where $0 \le a \le q-1$ and $m_a = a+1$ or $q-a$.

Next we consider the characters $R_{w_0}(\mu)$. Arguing as in the proof of Lemma 4.2, we obtain

LEMMA 4.3: *Each $R_{w_0}(\mu)$ with μ regular is equal to exactly one $R_{w_0}(a, b)$ with (a, b) satisfying*

$$(17) \qquad 1 \le a, b, a+2b \le q+1, \quad 2a+b \le q.$$

Let $d = \gcd(3, q+1)$. Then it is clear that each of $((q+1)(q-2)+(3-d))/6$ characters $\chi^{(u,v)}_{(q-1)(q^2-q+1)}$ is equal to one of $R_{w_0}(a, b)$ with (a, b) satisfying (17). It is not difficult to show that $R_{w_0}(0, 0) = St - 2\chi_{q(q-1)} - 1_G$,

$$R_{w_0}\left(\frac{q+1}{3}, \frac{q+1}{3}\right) = \sum_{u=0}^{2} \chi^{(u)}_{(q-1)(q^2-q+1)/3}$$

if $d = 3$, and

$$R_{w_0}(a, 0) = \chi^{(n_a)}_{q(q^2-q+1)} - \chi^{(n_a)}_{q^2-q+1},$$

where $0 \le a \le q-1$ and $n_a = a$ or $q-a+1$.

Finally, we consider the characters $R_{s_1}(\mu)$. Following the proof of Lemma 4.2, we obtain

LEMMA 4.4: *Each $R_{s_1}(\mu)$ with μ regular is equal to exactly one $R_{s_1}(r, s)$ with (r, s) satisfying one of the following three conditions:*

$$(18) \qquad \begin{array}{ll} \text{(a)} & r = 0,\ 1 \le s \le q-2, \\ \text{(b)} & 1 \le r \le q-1,\ 0 \le s \le q-2,\ 2r+s \le q-1,\ r+2s \le q-1, \\ \text{(c)} & 4 \le r \le q-1,\ 2 \le s \le q-3,\ 2r+s \ge 2q+1,\ r+2s \ge 2q-2. \end{array}$$

Each character $\chi^{(u)}_{(q+1)(q^2-1)}$ is equal to exactly one $R_{s_1}(r,s)$ with (r,s) satisfying (18). Furthermore, it is not difficult to show that $R_{s_1}(0,0) = St + \chi_{q(q-1)} - 1_G$, and (if $d=3$)

$$\left\{R_{s_1}\left(\frac{q+1}{3}, \frac{q-2}{3}\right), R_{s_2}\left(\frac{q-2}{3}, \frac{q+1}{3}\right)\right\} = \left\{\sum_{u=0}^{2} \chi^{(u,1)}_{(q+1)(q^2-1)/3}, \sum_{u=0}^{2} \chi^{(u,2)}_{(q+1)(q^2-1)/3}\right\}.$$

4.2 A CRITERION FOR GLOBAL IRREDUCIBILITY. Throughout this section we suppose that $G = \mathcal{G}^F = \mathrm{SU}_3(q)$, $q = p^f > 2$, and that *a nontrivial irreducible character $\chi \in \mathrm{Irr}(G)$ leads to a GIR*. By Proposition 1.2, for any arbitrary prime r all irreducible constituents of χ mod r are conjugate (over $\overline{\mathbb{F}_r}$ and under $\mathrm{Aut}(G)$). In particular, all irreducible constituents of χ mod r have the same degree, which divides $\deg\chi$. Every simple kG-module is isomorphic to exactly one $L(\lambda)$ with $\lambda = (r,s) \in X_f(T)$ (cf. §2), and the corresponding Brauer character is denoted by $\varphi(\lambda) = \varphi(r,s)$. The relation between $\varphi(\lambda)$ and $\zeta(\lambda)$ has been explained in §2.

COROLLARY 4.5: *Suppose that $\varphi(\lambda)$, $\varphi(\lambda')$ are irreducible constituents of $\chi \bmod p$, where $\lambda = (r,s)$ and $\lambda' = (r',s')$ belong to $X_f(T)$. Then $r+s \equiv r'+s' \bmod (p-1)$.*

Proof: Consider the p-adic decompositions of the integers r, s: $r = \sum_{i=0}^{f-1} p^i r_i$, $s = \sum_{i=0}^{f-1} p^i s_i$. By Lemma 4.1 [Tiep 3], the action of $\mathrm{Out}(G)$ on $\mathrm{IBr}_p(G)$ is generated by the map $\varphi(\lambda) \mapsto \varphi(\Psi(\lambda)) = (\tilde{r}, \tilde{s})$, where

$$\tilde{r} = s_{f-1} + \sum_{i=0}^{f-2} p^{i+1} r_i, \quad \tilde{s} = r_{f-1} + \sum_{i=0}^{f-2} p^{i+1} s_i.$$

In particular,

$$\tilde{r} + \tilde{s} = \sum_{i=0}^{f-2} \left(p^{i+1}(r_i + s_i)\right) + (s_{f-1} + r_{f-1}) \equiv r + s \bmod (p-1).$$

Therefore our claim follows from Proposition 1.2. ∎

4.3 REDUCTION MODULO p. Recall we are assuming that a certain irreducible complex character χ of G leads to a GIR of $H \rhd G$. It is well known that $St \bmod p$ (and of course $1_G \bmod p$) belongs to $\mathrm{IBr}_p(G)$. In this subsection, using reduction modulo p (the defining characteristic), we show that there are only a few possibilities for χ. A crucial ingredient of our arguments is Jantzen's formula for reduction modulo p of Deligne–Lusztig characters (cf. [Jan 1]), which states in the case $G = \mathrm{SU}_3(q)$ that

$$R_w(\mu) \bmod p = \sum_{u \in W} \zeta(u(\mu - w\pi\epsilon_{w_0 u}) + q\rho_u - \rho), \tag{19}$$

where

$$\rho_u = \sum_{\alpha \in S, u^{-1}(\alpha)<0} \omega_\alpha, \quad \epsilon_u = u^{-1}\rho_u.$$

In what follows we shall write $R_w(\mu) \bmod p$ as a sum $\sum_{i=1}^{6} \zeta(\lambda^i)$ and $\lambda^i = (r^i, s^i)$.

LEMMA 4.6: $\chi \neq \chi^{(u)}_{q^3+1}$.

Proof: Assume the contrary: $\chi = \chi^{(u)}_{q^3+1}$ for some u. Then, by Lemma 4.2, χ is equal to $R_1(r,s)$ with some (r,s) satisfying (15). According to (19), $\chi \bmod p = \sum_{i=1}^{6} \zeta(\lambda^i)$ with $\lambda^1 = (r,s)$, $\lambda^2 = (q-r-2, r+s)$, $\lambda^3 = (r+s, q-s-2)$, $\lambda^4 = (q-r-s-1, r-2)$, $\lambda^5 = (s-2, q-r-s-1)$, $\lambda^6 = (q-s-1, q-r-1)$. Clearly, $\lambda^i \in X_f(T)$ for $i = 1,2,3,6$, or for $i = 4$ and $r \geq 2$, or for $i = 5$ and $s \geq 2$. By Lemma 2.3, $\zeta(\lambda^4)$ is equal to 0 if $r = 1$ and $-\zeta(\lambda'^4)$ if $r = 0$, where $\lambda'^4 = (q-s-2, 0) \in X_f(T)$. Similarly, $\zeta(\lambda^5)$ is equal to 0 if $s = 1$ and $-\zeta(\lambda'^5)$ if $s = 0$, where $\lambda'^5 = (0, q-r-2) \in X_f(T)$.

1) First suppose that $r \geq 2$ and $s \geq 1$. Then it is clear that $\chi \bmod p$ contains $\varphi(r^i, s^i)$ with $i = 1,2,3,4,6$. By Corollary 4.5 we have $r+s \equiv r^i + s^i \bmod (p-1)$ for $i \neq 5$. This implies $p = 2$. In this case $\chi(1) = q^3+1$ is odd, hence each $\deg\varphi(r^i, s^i)$, being a divisor of $\chi(1)$, must be odd for $i \neq 5$. Now using Lemma 2.5 and (5) we see that $\deg\varphi(a,b)$ is equal to 1 if $(a,b) = (0,0)$, 3 if $\{a,b\} = \{0,1\}$, and 8 if $(a,b) = (1,1)$. By Steinberg's tensor product theorem the oddness of $\deg\varphi(r^i, s^i)$ forces $(r^i, s^i) \not\equiv (1,1) \bmod 2$. But one easily checks that this condition cannot be simultaneously satisfied by $i = 1,2,3,6$, a contradiction. Similarly, the case where $r \geq 1$ and $s \geq 2$ is impossible.

2) Now we consider the case where $r = 0$ and $1 \leq s \leq q-2$. Observe that $|\lambda^i|_2 > |\lambda'^4|_2$ for $i = 2,3,6$. Hence by Corollary 2.2 $\chi \bmod p$ contains $\varphi(\lambda^i)$ for all $i = 2,3,6$. Applying Corollary 4.5 one sees that $r^i + s^i \bmod (p-1)$ are the same for these indices i, therefore $p = 2$. But in this case for at least one index $i \in \{2,3,6\}$ we have $r^i \equiv s^i \equiv 1 \bmod 2$ and so $\deg\varphi(\lambda^i)$ is divisible by 8, again a contradiction. Similarly, the case $s = 0$ is impossible.

3) We have shown that $r = s = 1$. In this case $q \geq 4$ and $\chi \bmod p = \zeta(1,1) + \zeta(q-3,2) + \zeta(2,q-3) + \zeta(q-2,q-2)$. By Corollary 4.5, $p-1$ divides

2, i.e. $p=2$ or 3. The case $p=2$ cannot occur since $\chi \bmod p$ contains $\varphi(1,1)$. If $p=3$, then $\chi \bmod p$ contains $\varphi(q-3,2)$, and $\deg\varphi(q-3,2)$ is divisible by $\deg\varphi(0,2)=6$, while $\chi(1)=q^3+1$ is not divisible by 3. This final contradiction completes the proof of Lemma 4.6. ∎

Using Lemma 4.3 and following the proof of Lemma 3.5, we obtain

LEMMA 4.7: $\chi \neq \chi^{(u,v)}_{(q-1)(q^2-q+1)}$.

LEMMA 4.8: $\chi \neq \chi^{(u)}_{(q+1)(q^2-1)}$.

Proof: Assume the contrary: $\chi = \chi^{(u)}_{(q-1)(q^2-q+1)}$ for some u. Then, by Lemma 4.4, χ is equal to $R_{s_1}(r,s)$ with some (r,s) satisfying (18).

1) First we deal with the case of (18)(a). According to (19), $\chi \bmod p = \sum_{i=1}^{6}\zeta(\lambda^i)$ with $\lambda^1=(-2,s+1)$, $\lambda^2=(q,s-1)$, $\lambda^3=(s,q-s-2)$, $\lambda^4=(q-s-2,0)$, $\lambda^5=(s-1,q-s)$, $\lambda^6=(q-s-1,q-1)$. Clearly, $\lambda^i\in X(T)_+$ for $i>1$. Furthermore, by Lemma 2.3, $\zeta(\lambda^1)=-\zeta(\lambda'^1)$, where $\lambda'^1=(0,s)\in X_f(T)$. Since $|\lambda'^1|_1<|\lambda^i|_1$ for $i=3,5,6$, by Corollary 2.2, $\chi \bmod p$ contains $\varphi(\lambda^i)$ with $i=3,5,6$. Applying Corollary 4.5, we obtain $0\equiv 1 \bmod (p-1)$, i.e. $p=2$ (and $q\geq 4$). In this case $\chi(1)$ is odd, but either $\deg\varphi(\lambda^3)$ or $\deg\varphi(\lambda^6)$ is divisible by $\deg\varphi(1,1)=8$, a contradiction.

2) Next we consider the case of (18)(b). According to (19), $\chi \bmod p = \sum_{i=1}^{6}\zeta(\lambda^i)$ with $\lambda^1=(r-2,s+1)$, $\lambda^2=(q-r,r+s-1)$, $\lambda^3=(r+s,q-s-2)$, $\lambda^4=(q-r-s-2,r)$, $\lambda^5=(s-1,q-r-s)$, $\lambda^6=(q-s-1,q-r-1)$. Clearly, $\lambda^i\in X_f(T)$ for $i\neq 1,5$. If $i=1,5$ then, by Corollary 2.3, either $\zeta(\lambda^i)=0$ or $\lambda^i\in X_f(T)$. In particular, $\chi \bmod p$ contains $\varphi(\lambda^i)$ with $i=2,3,4,6$. Applying Corollary 4.5, we obtain $0\equiv 1 \bmod (p-1)$, i.e. $p=2$ (and $q\geq 4$). In this case $\chi(1)$ is odd, but at least one of $\varphi(\lambda^i)$, $i=2,3,4,6$, has degree divisible by $\deg\varphi(1,1)=8$, again a contradiction.

3) Finally, we handle the case of (18)(c). Then $\chi=R_{s_2}(r,s)$ with $1\leq r\leq q-4$, $2\leq s\leq q-3$, $2r+s\leq q-1$, $r+2s\leq q-1$. According to (19), $\chi \bmod p = \sum_{i=1}^{6}\zeta(\lambda^i)$ with $\lambda^1=(r+1,s-2)$, $\lambda^2=(q-r-2,r+s)$, $\lambda^3=(r+s-1,q-s)$, $\lambda^4=(q-r-s,r-1)$, $\lambda^5=(s,q-r-s-2)$, $\lambda^6=(q-s-1,q-r-1)$. Clearly, $\lambda^i\in X_f(T)$ for any i, and so $\chi \bmod p$ contains $\varphi(\lambda^i)$. Applying Corollary 4.5, we obtain $0\equiv 1 \bmod (p-1)$, i.e. $p=2$ (and $q\geq 4$). In this case $\chi(1)$ is odd, but at least one of $\varphi(\lambda^i)$, $i=1,2,3,6$, has degree divisible by $\deg\varphi(1,1)=8$. This contradiction completes the proof of Lemma 4.8. ∎

LEMMA 4.9: *$\chi \neq \chi^{(u)}_{q(q^2-q+1)}, \chi^{(u)}_{q^2-q+1}$, unless $q = p = 3$ and $\chi = \chi^{(2)}_{q^2-q+1}$.*

Proof: 1) First assume that $\chi = \chi^{(u)}_{q(q^2-q+1)}$ for some u. Then, by the results of §4.1 and (19),

$$\chi \bmod p = \tfrac{1}{2}(R_{w_0}(s,0) + R_1(q-s,s-1)) \bmod p$$
$$= \zeta(q-s,s-1) + \zeta(s-2,q-1) + \zeta(q-s-1,s-2) + \zeta(q-1,q-s-1)$$

for some s, $1 \leq s \leq q$. If $s = 1$, then $\chi \bmod p = \zeta(q-1,0) + \zeta(q-1,q-2)$. In particular, $\chi \bmod p$ contains $\varphi(q-1,0)$, whose degree is $\left(\frac{p(p+1)}{2}\right)^f$, and $\varphi(q-1,q-2)$, whose degree is $\frac{p(p-1)(2p-1)}{2}p^{3(f-1)}$. By Proposition 1.2, $\deg\varphi(q-1,0) = \deg\varphi(q-1,q-2)$, hence $q = 2$, contrary to our assumption. The case $s = q$ can be excluded in the same way.

Thus we can suppose that $2 \leq s \leq q-1$. In this case $\chi \bmod p = \sum_{i=1}^{4}\zeta(\lambda^i)$ with $\lambda^1 = (q-s,s-1)$, $\lambda^2 = (s-2,q-1)$, $\lambda^3 = (q-1,q-s-1)$, $\lambda^4 = (q-s-1,s-2)$. Clearly, $\lambda^i \in X_f(T)$ for any i, and so χ mod p contains $\varphi(\lambda^i)$. Applying Corollary 4.5, we obtain $s \equiv 0 \bmod (p-1)$ and $(p-1)|2$, i.e. $p = 2,3$. If $p = 2$ then $\deg\varphi(\lambda^1) = 3^f$, while either $\deg\varphi(\lambda^2)$ or $\deg\varphi(\lambda^3)$ is divisible by $\deg\varphi(1,1) = 8$, contrary to Proposition 1.2. It remains to consider the case $p = 3$. If $s \neq (q+1)/2$ then $\deg\varphi(\lambda^1)$ is divisible by $\deg\varphi(0,2) = 6$, while $\chi(1)$ is odd, a contradiction. If $s = (q+1)/2$, then

$$\lambda^1 = \left(\frac{q-1}{2}, \frac{q-1}{2}\right)$$

is invariant under $\mathrm{Aut}(G)$ and Fr. In particular, $\varphi(\lambda^3)$ is not conjugate to $\varphi(\lambda^1)$, contrary to Proposition 1.2.

2) Next we assume that $\chi = \chi^{(u)}_{q^2-q+1}$ for some u. Then χ is an irreducible Weil character of G, and hence our statement follows from §4 of [Tiep 3]. ∎

LEMMA 4.10: *If $\chi = \chi_{q(q-1)}$, then $q = p$ or $q = p^2$.*

Proof: Again, $\chi_{q(q-1)}$ is an irreducible Weil character of $G = \mathrm{SU}_3(q)$. Hence our statement follows from §4 of [Tiep 3]. ∎

Following the proof of Lemma 3.11, we obtain

LEMMA 4.11: *Suppose that $q \equiv -1 \pmod 3$ (and $q > 2$). Then*

$$\chi \neq \chi^{(u)}_{(q-1)(q^2-q+1)/3}, \quad \chi^{(u,v)}_{(q+1)(q^2-1)/3}.$$

COROLLARY 4.12: *Let* $\chi \in \mathrm{Irr}(G)$, *where* $G = \mathrm{SU}_3(q)$ *and* $q = p^f > 2$. *Then* $\chi \bmod p$ *is absolutely irreducible if and only if either* $\chi \in \{1_G, St\}$, *or* $q = 3$ *and* $\chi = \chi^{(2)}_{q^2-q+1}$.

4.4 PROOF OF THEOREM 1.4. Assume that G is a covering group of $L = \mathrm{PSU}_3(q)$, $q = p^f$, and that a nontrivial character $\chi \in \mathrm{Irr}(G)$ leads to a GIR of H, $H \rhd G$. Since $\mathrm{Mult}(L) = \mathbb{Z}_{\gcd(3,q+1)}$, it is clear that G is a factor group of $\mathrm{SU}_3(q)$, and so without loss of generality one can suppose that $G = \mathrm{SU}_3(q)$. By the results of §4.3, either $f \leq 2$ and $\chi = \chi_{q(q-1)}$, or $\chi = St$, or $q = 3$ and $\chi = \chi^{(2)}_{q^2-q+1}$. The first possibility can in fact be realized, and the corresponding global irreducibles were constructed by Gross in [Gro]; see the discussion in §1 after Theorem 1.4. The second case is impossible since there always exists an odd prime ℓ such that $St \bmod \ell$ contains the trivial character (cf. [Geck]). Finally, the third case cannot occur, since $\chi^{(2)}_{q^2-q+1} \bmod 2$ contains the trivial character when $q = 3$ (cf. [ATLAS]).

Theorem 1.4 has now been completely proved. ∎

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